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Analytic Spectral Integration of Birefringence-Induced Iridescence - Supplemental Material

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1. Optical Anisotropy

In this section an overview of the physical theory of light propagating in an optically anisotropic medium is provided. We consider two homogeneous and magnetically isotropic adjacent media with their interface forming the XZ-plane in our coordinate system of choice and $I_i = \left[0, \cos\theta, \sin\theta\right]^T$ being a ray propagating from the source medium ($Y > 0$) to the destination medium ($Y < 0$) with the incidence point being at the origin (see figure 2 in the paper). The incidence angle of I_i is then θ and we denote η_i as the effective index-of-refraction as perceived by the incident wave propagating in the source medium, which for isotropic materials is the refractive index of the source medium.

Let $\mathbf{E} = e^{i(qy-Kz-\omega t)}[\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z]^T$ and $\mathbf{B}$ be the electric and magnetic field vectors, with ω being the angular frequency, q the component of the wave's propagation direction that is normal to the surface and K, the *incidence parameter*, which is the tangential component of the incident wave vector:

$$K = k\eta_i \sin(\theta) \tag{1}$$

Here $k = \frac{\omega}{v_p}$ is the wavenumber with v_p being the wave's phase velocity. The curl equations of Maxwell are then

$$\nabla \times \mathbf{E} = ik\mathbf{B} \qquad \nabla \times \mathbf{B} = -ik\mathbf{D} = -ik\hat{\varepsilon}\mathbf{E} \tag{2}$$

with $\hat{\varepsilon}$ being the permittivity and $\mathbf{D}$ the displacement field. The permittivity is generally a complex value and can be written as $\hat{\varepsilon} = \varepsilon + i\sigma$, where the real part is the dielectric constant and the imaginary part is the conductivity. For non-conducting media $\sigma \equiv 0$ and $\hat{\varepsilon} \equiv \varepsilon$. While $\hat{\varepsilon}$ is a (possibly complex) scalar for isotropic media, in the case of optical anisotropy $\hat{\varepsilon}$ becomes a symmetric rank 2 tensor [BW99] describing the linear relation between the displacement field $\mathbf{D}$ and the electric field $\mathbf{E}$, which now have different directions.

As the dielectric tensor ε and the conductivity tensor σ are real symmetric matrices, by the spectral theorem they can be diagonalized by some orthogonal matrices. We restrict our discussion to materials where both tensors are diagonalize-able by a single orthogonal matrix Q, that is

$$Q^T\hat{\varepsilon}Q = \begin{bmatrix} \hat{\varepsilon}_x & 0 & 0 \\ 0 & \hat{\varepsilon}_y & 0 \\ 0 & 0 & \hat{\varepsilon}_z \end{bmatrix} = \begin{bmatrix} \varepsilon_x + i\sigma_x & 0 & 0 \\ 0 & \varepsilon_y + i\sigma_y & 0 \\ 0 & 0 & \varepsilon_z + i\sigma_z \end{bmatrix} \tag{3}$$

where $\varepsilon_{x,y,z}$ and $\sigma_{x,y,z}$ are the *principal dielectric constants* and *principal conductivity*, respectively. Anisotropic materials are generally classified into two categories: *Uniaxial*, where $\hat{\varepsilon}_x = \hat{\varepsilon}_y \neq \hat{\varepsilon}_z$, and *biaxial*, where $\hat{\varepsilon}_x \neq \hat{\varepsilon}_y \neq \hat{\varepsilon}_z \neq \hat{\varepsilon}_x$. The derivations for biaxial materials is rather complex, therefore we limit ourselves to uniaxial materials and denote $\hat{\varepsilon}_o = \hat{\varepsilon}_x$ as the *ordinary* permittivity and $\hat{\varepsilon}_e = \hat{\varepsilon}_z$ as the *extraordinary* permittivity. Then the ordinary and extraordinary indices-of-refraction are $\eta_o = \sqrt{\hat{\varepsilon}_o}$ and $\eta_e = \sqrt{\hat{\varepsilon}_e}$, respectively. As we will see later, a wave propagating in an uniaxial medium perceives either the ordinary index-of-refraction and is then an ordinary wave, or an index-of-refraction between the ordinary and the extraordinary ones, depending on its polarization in relation to the optic axis, and is an extraordinary wave. The term "extraordinary" refers to the fact that an extraordinary wave propagates in a remarkable manner unseen in isotropic optics, e.g. it violates the law of reflection and Snell's law of refraction.

The optic axis Notice that as $\hat{\varepsilon}_e$ is distinct from the other two principal permittivity constants, the optical properties of an uniaxial medium give rise to a distinguished direction, termed the *optic axis*, which is the direction of the Z axis of the principal coordinate frame, i.e. the coordinate system under which the permittivity tensor $\hat{\varepsilon}$ is diagonalized. $\hat{\varepsilon}$ is then defined uniquely by the optic axis and the ordinary and extraordinary indices of refraction: We denote the optic axis as $A = [\alpha, \beta, \gamma]^T$ and write Q as

$$Q = \begin{bmatrix} a_1 & b_1 & \alpha \\ a_2 & b_2 & \beta \\ a_3 & b_3 & \gamma \end{bmatrix} \tag{4}$$

then, using equation 3, we can write the permittivity tensor as

$$\hat{\varepsilon} = Q \begin{bmatrix} \hat{\varepsilon}_o & 0 & 0 \\ 0 & \hat{\varepsilon}_o & 0 \\ 0 & 0 & \hat{\varepsilon}_e \end{bmatrix} Q^T =$$

$$= \begin{bmatrix} (a_1^2+b_1^2)\hat{\varepsilon}_o+\alpha^2\hat{\varepsilon}_e & (a_1a_2+b_1b_2)\hat{\varepsilon}_o+\alpha\beta\hat{\varepsilon}_e & (a_1a_3+b_1b_3)\hat{\varepsilon}_o+\alpha\gamma\hat{\varepsilon}_e \\ (a_1a_2+b_1b_2)\hat{\varepsilon}_o+\alpha\beta\hat{\varepsilon}_e & (a_2^2+b_2^2)\hat{\varepsilon}_o+\beta^2\hat{\varepsilon}_e & (a_2a_3+b_2b_3)\hat{\varepsilon}_o+\beta\gamma\hat{\varepsilon}_e \\ (a_1a_3+b_1b_3)\hat{\varepsilon}_o+\alpha\gamma\hat{\varepsilon}_e & (a_2a_3+b_2b_3)\hat{\varepsilon}_o+\beta\gamma\hat{\varepsilon}_e & (a_3^2+b_3^2)\hat{\varepsilon}_o+\gamma^2\hat{\varepsilon}_e \end{bmatrix} \tag{5}$$

and using the fact that Q is orthogonal the permittivity tensor reduces to

$$\hat{\varepsilon} = \begin{bmatrix} \hat{\varepsilon}_o + \alpha^2\Delta\varepsilon & \alpha\beta\Delta\varepsilon & \alpha\gamma\Delta\varepsilon \\ \alpha\beta\Delta\varepsilon & \hat{\varepsilon}_o + \beta^2\Delta\varepsilon & \beta\gamma\Delta\varepsilon \\ \alpha\gamma\Delta\varepsilon & \beta\gamma\Delta\varepsilon & \hat{\varepsilon}_o + \gamma^2\Delta\varepsilon \end{bmatrix} = \hat{\varepsilon}_o I + \Delta\varepsilon A A^T \tag{6}$$

with $\Delta\varepsilon = \hat{\varepsilon}_e - \hat{\varepsilon}_o$. The real part of $\Delta\varepsilon$ is the material's *degree of anisotropy*. Positive degrees of anisotropy indicates that the ordinary wave propagates faster than the extraordinary wave while negative values indicate the opposite.

It is worth noting that the permittivity, and in turn the indices-of-refraction and the optic axis, generally varies with the wave's frequency, temperature and other factors, therefore can be treated as a constant only for a given monochromatic wave.

Sources of Optical Anisotropy We briefly discuss the common sources of optical anisotropy: *Crystals* are naturally anisotropic due to the anisotropic electrical properties of the molecules that compose their crystal lattices. The lattice system of a crystal induces different types of optical properties and anisotropy: Crystals can be isotropic (e.g. Cubic crystals), uniaxial (e.g. Hexagonal crystals) or biaxial (e.g. Monoclinic crystals), furthermore some crystals exhibit a phenomena called *dispersion of axes* [BW99], where the optic axes shift depending on the wavelength. *Form anisotropy* is optical anisotropy induced in otherwise isotropic materials by structural features that are larger than molecules but smaller than the light's coherence size, e.g. a material composed of particles forming thin parallel sheets will be optically anisotropic. The *Photoelasticity effect* is facilitated by mechanical stress applied to an isotropic material that alters the permittivity tensor resulting in anisotropy. Photoelasticity is of particular interest due to its prevalence (e.g. a window with stress applied by the frame) as well as practical applications.

1.1. Derivation of the Electric Fields and Poynting Vectors

Simplifying equations 2 for an homogeneous anisotropic medium where $\partial\mathbf{E}/\partial x = \mathbf{0}$ gives rise to the following relation between the displacement and electric fields [Lek91]:

$$k^2\mathbf{D} = \nabla\times(\nabla\times\mathbf{E}) = \begin{bmatrix} (q^2+K^2)\mathbf{E}_x \\ K^2\mathbf{E}_y + qK\mathbf{E}_z \\ qK\mathbf{E}_y + q^2\mathbf{E}_z \end{bmatrix} e^{i(qy-Kz-\omega t)} \tag{7}$$

The electric field can now be expressed in terms of the medium's optical properties, the permittivity tensor as well as the optic axis, by substituting equation 6 into 2 and together with 7 we get the following system for the normal modes (fields propagating as plane waves in the medium):

$$\hat{\varepsilon}\mathbf{E} - \mathbf{D} = \left(\hat{\varepsilon} - \frac{1}{k^2}\begin{bmatrix} q^2+K^2 & 0 & 0 \\ 0 & K^2 & qK \\ 0 & qK & q^2 \end{bmatrix}\right)\cdot\mathbf{E} = M\mathbf{E} = 0 \tag{8}$$

Solving 8 for $\mathbf{E}$ is equivalent to finding the eigenspace of M associated with eigenvalue 0, that is solving $|M| = 0$ for q. This gives rise to four normal modes:

$$\begin{aligned} q_o^{\pm} &= \pm\sqrt{\hat{\varepsilon}_o k^2 - K^2} \\ q_e^{\pm} &= \frac{\pm\sqrt{\hat{\varepsilon}_o}\sqrt{\hat{\varepsilon}_e k^2\left(\beta^2\Delta\varepsilon + \hat{\varepsilon}_o\right) + K^2\left(\alpha^2\Delta\varepsilon - \hat{\varepsilon}_e\right)} + \beta\gamma K\Delta\varepsilon}{\hat{\varepsilon}_o + \beta^2\Delta\varepsilon} \end{aligned} \tag{9}$$

As q is the wave vector's normal component, normal modes superscripted with a plus correspond to waves travelling upwards while normal modes with a minus to waves travelling downwards, in other words q^+ correspond to reflections and q^- to refractions. The ordinary and extraordinary wave's electric fields are the eigenvectors of M corresponding to the respective normal modes:

$$\begin{aligned} \mathbf{E}_o^{\pm} &= \frac{1}{N_o^{\pm}}\left[-\beta K - \gamma q_o^{\pm}, \alpha K, \alpha q_o^{\pm}\right]^T \\ \mathbf{E}_e^{\pm} &= \frac{1}{N_e^{\pm}}\begin{bmatrix} \alpha k^2\hat{\varepsilon}_o \\ \beta k^2\hat{\varepsilon}_o + q_e^{\pm}\left(\gamma K - \beta q_e^{\pm}\right) \\ \gamma k^2\hat{\varepsilon}_o - K\left(\gamma K - \beta q_e^{\pm}\right) \end{bmatrix} \end{aligned} \tag{10}$$

where N are normalization factors. The directions of propagation of the waves' associated with each normal mode are

$$\mathbf{W}_{o,e}^{\pm} = \frac{\left[0, q_{o,e}^{\pm}, -K\right]^T}{\sqrt{\left(q_{o,e}^{\pm}\right)^2 + K^2}} \tag{11}$$

The ordinary and extraordinary ray directions are defined by the direction of their *Poynting vectors*, which represent the power and flow direction of the electromagnetic flux. The Poynting vector is then [BW99]:

$$\bar{S} = \frac{1}{\mu}\mathbf{E}\times\mathbf{B} = \frac{-i}{k\mu}\mathbf{E}\times(\nabla\times\mathbf{E}) = \frac{1}{k\mu}\mathbf{E}\times(\mathbf{W}\times\mathbf{E}) \tag{12}$$

where μ is the permeability, which is a scalar by our assumption of isotropically magnetic media, resulting in ray direction $I = \bar{S}/\left|\bar{S}\right|$. We can now deduce a few theoretical conclusions:

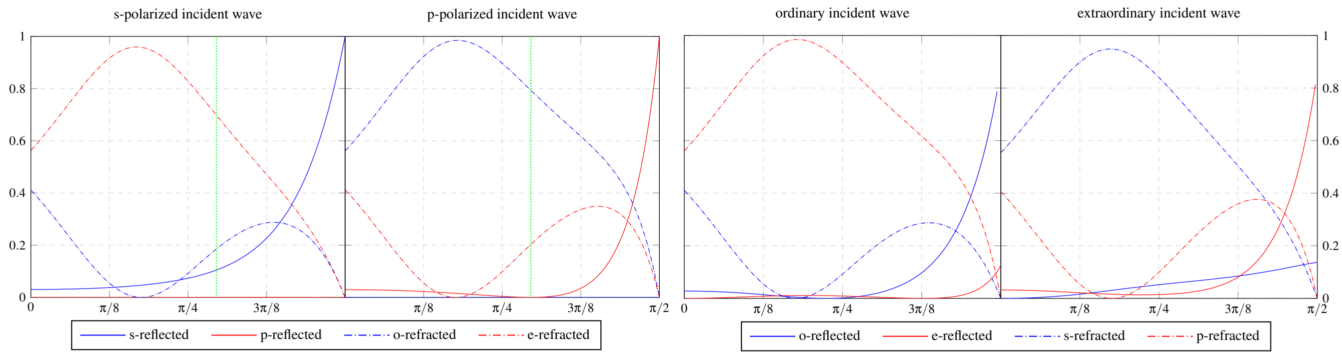
1. The ordinary wave is always polarized perpendicular to the optic axis as $\langle\mathbf{E}_o, A\rangle = 0$ ($\langle\cdot\rangle$ denotes the inner product).
2. While an extraordinary ray is generally detached from its wave direction, an ordinary ray coincides with its wave's direction of propagation, $\mathbf{W}_o$. Therefore, ordinary ray direction is simply $I_o = \mathbf{W}_o$ and obeys the law of reflection for reflections and Snell's law for refractions.
3. The ordinary ray always remains in the plane of incidence, as $(I_o)_x \equiv 0$. That is however generally not true for the extraordinary ray.
4. In the isotropic limit ($\Delta\varepsilon\to 0$) we obtain that $q_o^{\pm}\equiv q_e^{\pm}$ and $\langle\mathbf{E}_o^{\pm}, \mathbf{E}_e^{\pm}\rangle = 0$, that is the extraordinary and ordinary rays coincide and are polarized perpendicular to each other.

To compute the ray directions of ordinary and extraordinary reflected rays the optic axis and dielectric constants of the incidence media (Y>0) are used to derive q_o^+, q_e^+ via equation 9 and the corresponding electric fields $\mathbf{E}_o^+$, $\mathbf{E}_e^+$ using equation 10, while for a refracted ray we use the optic axis and dielectric constants of the refraction media (Y<0) to compute q_o^-, q_e^-, $\mathbf{E}_o^-$, $\mathbf{E}_e^-$. Remember that the incidence angle is the associated wave's direction of propagation (equation 11) incidence angle. The normal modes and electric fields can then be used to compute the Poynting vectors via equation 12.

As the incidence parameter K remains constant for all participating waves, the effective (real) refractive index perceived by a wave propagating in an anisotropic media is [YY03]

$$\eta_{ef} = \frac{K}{k\sin\phi} = \frac{1}{k}\sqrt{q^2+K^2} \tag{13}$$

where ϕ is wave's angle of refraction or reflection. And indeed while for ordinary waves η_{ef} remains fixed at η_o, extraordinary waves perceive a refractive index between η_o and η_e. The ray's



(a) Isotropic-to-anisotropic interface

(b) Anisotropic-to-isotropic interface

Figure 1: Plots of Fresnel power ratios at isotropic-anisotropic interfaces. The isotropic medium's index-of-refraction is fixed at $\eta = 1$ while the anisotropic medium's constants are: $\eta_o = 1.4$, $\eta_e = 1.55$, $A = [0.1, 0.9, 0.42]^T$. The Brewster's angle, the angle at which the reflected light is fully s-polarized, is marked with a dotted green line and varies slightly as a function of the optic axis [SP91].

phase velocity in direction of the Poynting vector I is computed by projecting η_{ef} onto I:

$$v_p = \frac{c}{\langle \mathbf{W}, I \rangle \cdot \eta_{ef}} \tag{14}$$

where $\langle \mathbf{W}, I \rangle \cdot \eta_{ef}$ is the refractive index perceived by the ray. From 14 it is evident that an uniaxial material acts as a retarder. The *maximal incidence parameter*, closely related to the critical angle, is defined as

$$K_e^{max} = k\eta_e \sqrt{\frac{\eta_o^2 + \beta^2 \Delta\varepsilon}{\eta_e^2 - \gamma^2 \Delta\varepsilon}} \qquad K_o^{max} = k\eta_o \tag{15}$$

When $K \geq K^{max}$ total-reflection occurs: The refracted wave becomes evanescent and its associated Poynting vector is complex and parallel to the surface interface.

A note regarding k The wavenumber k is a factor of the incidence parameter K (equation 1), therefore the normal modes q (equations 9) admit k as a factor as well. It is then easy to see that due to the normalization in equations 10 k vanishes from the electric field vectors, and similarly the wave direction of propagation (equation 11) and the Poynting vectors (equation 12) do not depend on k. Thus k can be omitted and the only wavelength dependence that arises in the electric fields and Poynting vectors is in the medium constants.

1.2. Conductivity and absorption

When the conductivity tensor σ is non-zero the material is conductive and therefore absorbing. In this case the indices-of-refraction are complex values and can be written as $\eta_o = \sqrt{\hat{\varepsilon}_o} = n_o + i\kappa_o$, $\eta_e = \sqrt{\hat{\varepsilon}_e} = n_e + i\kappa_e$, which in turn makes the phase velocities (equation 14) of the ordinary and extraordinary waves complex. As the wavenumber $k = \omega / v_p$ is present as a factor both in q and K (equations 1, 9) a factor $e^{-\kappa(y-z)}$, which is Beer-Lambert's extinction law with the extinction coefficient κ, appears in the electric field amplitude $\mathbf{E}$. κ is the imaginary part of the perceived index of refraction, that is η_o for ordinary waves and η_{ef} (equation 13) for extraordinary waves.

In other words, the extinction coefficient depends on the polarization direction of the wave [Blo61]: An uniaxial medium gives rise to two extinction coefficients, for light polarized parallel and perpendicular to the optic axis, $\kappa_{\parallel}$ and $\kappa_{\perp}$, respectively. And indeed both formulations are equivalent in practice: From equation 10 it is evident that $\langle \mathbf{E}_o, A \rangle = 0$, that is an ordinary ray is always linearly polarized perpendicular to the optic axis and perceives the extinction coefficient $\kappa_o = \kappa_{\perp}$, while an extraordinary ray's extinction coefficient is

$$\kappa_e = |\langle \mathbf{E}_e, A \rangle| (\kappa_{\parallel} - \kappa_{\perp}) + \kappa_{\perp} \tag{16}$$

If the conductivity (and therefore the extinction coefficients) vary non-uniformly with wavelength then light refracting through the material might appear coloured differently based on the orientation relative to the optic axis. In this case the material is described as *pleochroic*, or, specifically for uniaxial materials, *dichroic*, as up to two distinct colours may appear.

2. Fresnel Transmission and Reflection Coefficients

Given an interface between two media as well as an incident ray we would like to compute the reflection and transmission Fresnel coefficients. The participating electric fields at the incidence point are the incident, reflected and transmitted electric fields:

$$\begin{aligned}
\mathbf{E}_I &= e^{i\left(q_i^- y - Kz - \omega t\right)} \mathbf{E}_i^- \\
\mathbf{E}_R &= r_o e^{i\left(q_o^+ y - Kz - \omega t\right)} \mathbf{E}_o^+ + r_e e^{i\left(q_e^+ y - Kz - \omega t\right)} \mathbf{E}_e^+ \\
\mathbf{E}_T &= t_o e^{i\left(\hat{q}_o^- y - Kz - \omega t\right)} \hat{\mathbf{E}}_o^- + t_e e^{i\left(\hat{q}_e^- y - Kz - \omega t\right)} \hat{\mathbf{E}}_e^-
\end{aligned} \tag{17}$$

where $q_i^- < 0$, $\mathbf{E}_i^-$ is the incident electric field, q_o^+, $\mathbf{E}_o^+$ and q_e^+, $\mathbf{E}_e^+$ are two reflected electric fields, and $\hat{q}_o^-$, $\hat{\mathbf{E}}_o^-$ and $\hat{q}_e^-$, $\hat{\mathbf{E}}_e^-$ are the transmitted electric fields. The Fresnel coefficients are r_o, r_e, t_o and t_e.

The interface conditions implied by the Maxwell equations [BW99] state that the tangential components of the electric field $\mathbf{E}$ are continuous across the interface. Furthermore, if we assume that no surface charge is present then the normal component of the displacement field $\mathbf{D}$ as well as the tangential component of the magnetic vector $\mathbf{H}=\frac{1}{\mu}\mathbf{B}$ are also continuous across the interface. That is $\mathbf{E}_x$, $\mathbf{E}_z$, $k^2\mathbf{D}_y=K^2\mathbf{E}_y+qK\mathbf{E}_z$ (equation 7) and $\mathbf{B}_z=ik\partial\mathbf{E}_x/\partial y$ (equation 2) are continuous across the interface which gives rise to the following system of 4 equations with 4 unknowns:

$$\begin{aligned}(\mathbf{E}_T)_x &= (\mathbf{E}_I)_x+(\mathbf{E}_R)_x\\ (\mathbf{E}_T)_z &= (\mathbf{E}_I)_z+(\mathbf{E}_R)_z\\ \frac{\partial(\mathbf{E}_T)_z}{\partial y} &= \frac{\partial(\mathbf{E}_I)_z}{\partial y}+\frac{\partial(\mathbf{E}_R)_z}{\partial y}+iK\left[(\mathbf{E}_I)_y+(\mathbf{E}_R)_y-(\mathbf{E}_T)_y\right]\\ \frac{\partial(\mathbf{E}_T)_x}{\partial y} &= \frac{\partial(\mathbf{E}_I)_x}{\partial y}+\frac{\partial(\mathbf{E}_R)_x}{\partial y}\end{aligned} \tag{18}$$

Solving the system yields the Fresnel coefficients for the general case of anisotropic-to-anisotropic interfaces. When one of the media is an isotropic medium the results can be significantly simplified. The Fresnel coefficients are listed in full in the appendix of the paper for isotropic-to-anisotropic and anisotropic-to-isotropic interfaces. The anisotropic-to-anisotropic coefficients are also provided as MATLAB scripts.

The Fresnel coefficients are, in general, complex values of the form $Ee^{i\Delta\psi}$, $E\in\mathbb{R}$, meaning that a phase shift of $\Delta\psi$ is introduced. For perfect dielectrics the conductivity is zero therefore the permittivity is real and the Fresnel coefficients will be real valued scalars (when $K<K^{max}$), that is $\Delta\psi\in\{0,\pi\}$, however that does not hold in general.

The power of an electromagnetic wave is carried by both the electric field and magnetic field, therefore the intensity of a light wave is the amplitude squared. However, the change in impedance and direction of the reflected or refracted waves need to be taken into account. Therefore, the power ratios are

$$\begin{aligned}R_o &= \frac{\eta_o\cos\psi_o}{\eta_i\cos\theta}r_or_o^\star=-\frac{q_o^+}{q_i^-}r_or_o^\star\\ R_e &= \frac{\eta_{ef}\cos\psi_e}{\eta_i\cos\theta}r_er_e^\star=-\frac{q_e^+}{q_i^-}r_er_e^\star\\ T_o &= \frac{\hat\eta_o\cos\phi_o}{\eta_i\cos\theta}t_ot_o^\star=\frac{q_o^-}{q_i^-}t_ot_o^\star\\ T_e &= \frac{\hat\eta_{ef}\cos\phi_e}{\eta_i\cos\theta}t_et_e^\star=\frac{q_e^-}{q_i^-}t_et_e^\star\end{aligned} \tag{19}$$

where $\star$ denotes the complex conjugate, the index-of-refraction as perceived by the incident wave propagating in the source medium is η_i, η_o and $\hat\eta_o$ are the indices of refraction of the source medium and medium of refraction, respectively, η_{ef} and $\hat\eta_{ef}$ are the effective indices of refraction as perceived by the reflected and refracted extraordinary waves, respectively, computed via equation 13, θ is the incidence angle, ψ_o and ψ_e are the angles of reflection off the reflected ordinary and extraordinary waves, respectively, and ϕ_o and ϕ_e are the angles of refraction of the refracted ordinary and extraordinary waves, respectively. Remember that the angles are the wave's angles of incidence, refraction and reflection and not the Poynting vectors'. Note that for incident ordinary rays it always holds that $R_o=r_or_o^\star$ as there is no change in angle or impedance. The total reflectivity is then $\mathcal{R}=R_o+R_e$ and total transmissivity is $\mathcal{T}=T_o+T_e$ and due to energy conservation it always holds that $\mathcal{R}+\mathcal{T}=1$ (see figure 1 and the accompanying MATLAB scripts for the Fresnel coefficients).

3. Iridescence when Reflecting of an Anisotropic Slab

Just as birefringence introduces iridescence when refracting through an anisotropic slab, iridescence can be also introduced in an almost identical manner in the case of reflection (see figure 3 in the paper). The primary path is then the reflection off the surface, while the secondary paths are any path that is refracted into the surface and escapes upwards. For the case of reflection we use the transmission coefficients for the upper interface, t_{os}, t_{op}, t_{es} and t_{ep}, furthermore we slightly change notation of the order of a path, n, which now denotes the count of internal reflections plus one. The functions for the OPD as well as a are simplfied as $OPD_o\equiv 0$, $P_z\equiv 0$:

$$\Delta\mathcal{P}=\mathcal{P}-a\eta_i \tag{20}$$

$$a(z)=|z|\sin\theta \tag{21}$$

Computation of the Jones vector for the case of reflection is very similar, with a couple of differences. First, all paths now undergo a reflection off the bottom interface, therefore we define D_0^R as follows:

$$D_0^R=\begin{bmatrix}\hat r_{oo}e^{ik\Delta\mathcal{P}_{oo}^R} & \hat r_{oe}e^{ik\Delta\mathcal{P}_{oe}^R}\\ \hat r_{eo}e^{ik\Delta\mathcal{P}_{eo}^R} & \hat r_{ee}e^{ik\Delta\mathcal{P}_{ee}^R}\end{bmatrix} \tag{22}$$

where the OPDs have the reflection order switched – we first reflect of the upper interface and then of the lower interface. Second, the amplitudes of primary path, the path that reflects of the upper interface and never refracts into the surface, need to be taken into account as an additional Jones vector:

$$\mathcal{J}_{ref}=\begin{bmatrix}r_{ss}+r_{ps}\\ r_{sp}+r_{pp}\end{bmatrix}e^{i(k\mathbf{W}-\omega t)} \tag{23}$$

Then the expression for aggregated amplitudes of all paths reflecting of an anisotropic slab is:

$$\begin{aligned}\mathcal{J}^R &= \mathcal{J}_{ref}+\sum_{n=0}^{\infty}\mathcal{J}_n^R=\\ &= \mathcal{J}_{ref}+\begin{bmatrix}t_o\\ t_e\end{bmatrix}^T(I-\mathcal{D}^R)^{-1}D_0^R\begin{bmatrix}t_{os} & t_{op}\\ t_{es} & t_{ep}\end{bmatrix}e^{i(k\mathbf{W}-\omega t)}\end{aligned} \tag{24}$$

with $\mathcal{D}^R$ serving the same purpose as $\mathcal{D}$, except the order 1 paths now start at the upper interface:

$$\mathcal{D}^R=\begin{bmatrix}r_{oo}\hat r_{oo}e^{ik\Delta\mathcal{P}_{oo}^R}+r_{oe}\hat r_{eo}e^{ik\Delta\mathcal{P}_{eo}^R} & r_{oo}\hat r_{oe}e^{ik\Delta\mathcal{P}_{oe}^R}+r_{oe}\hat r_{ee}e^{ik\Delta\mathcal{P}_{ee}^R}\\ r_{eo}\hat r_{oo}e^{ik\Delta\mathcal{P}_{oo}^R}+r_{ee}\hat r_{eo}e^{ik\Delta\mathcal{P}_{eo}^R} & r_{eo}\hat r_{oe}e^{ik\Delta\mathcal{P}_{oe}^R}+r_{ee}\hat r_{ee}e^{ik\Delta\mathcal{P}_{ee}^R}\end{bmatrix} \tag{25}$$

3.1. OPDs for Reflection of an Anisotropic Slab

In the case of reflection off an anisotropic slab we simply switch the order, that is we reflect first of the upper interface and then of the lower interface. Then the OPDs used for D_0^R and $\mathcal{D}^R$ (equations

22 and 25) are:

$$\Delta\mathcal{P}_{oo}^{R} = \Delta\mathcal{P}_{oo} \qquad \Delta\mathcal{P}_{eo}^{R} = \Delta\mathcal{P}_{oe}$$
$$\Delta\mathcal{P}_{oe}^{R} = \Delta\mathcal{P}_{eo} \qquad \Delta\mathcal{P}_{ee}^{R} = \Delta\mathcal{P}_{ee} \tag{26}$$

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